Note on the Darboux transformation for the derivative nonlinear Schrodinger equation

1991 J. Phys. A: Math. Gen. 24 363

(http://iopscience.iop.org/0305-4470/24/1/041)

J. Phys. A: Math. Gen. **24** (1991) 363–366. Printed in the UK

COMMENT

Note on the Darboux transformation for the derivative nonlinear Schrödinger equation

Yi Xiao

Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

Received 16 May 1990, in final form 26 June 1990

Abstract. *The Darboux and Backlund transformations for the derivative nonlinear* Schrödinger equation are derived by the same method for the nonlinear Schrödinger equation.

Darboux transformation is a simple and powerful method for generating solutions to integrable nonlinear equations and has been investigated by many authors [1–5]. In [1] the equivalence between the 'adding' one-soliton Darboux transformation and the dressing method was proven and it was shown how the Darboux matrices (DM) are determined by projection matrices (PM). In [5], we presented a simple method for constructing the DM of the nonlinear Schrödinger equation in which the PM can be determined by simply solving differential equations resulting from the Lax equations. For the sake of convenience, we call it the PM method in the following. In a recent paper [6], in order to obtain the soliton solutions of the derivative nonlinear Schrödinger (DNLS) equation, the PM method is modified by Huang and Chen because the differential equations for the DM are hard to solve in this case. In this comment, we shall show that the PM method can still apply to the DNLS equation.

The DNLS equation is

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}(|u|^2u)_x = 0 \tag{1}$$

whose Lax equations are [6]

$$F_x(\zeta) = L(\zeta)F(\zeta) \tag{2a}$$

$$F_t(\zeta) = M(\zeta)F(\zeta) \tag{2b}$$

where

$$L(\zeta) = -\mathrm{i}\zeta^{-2}\sigma_3 + \zeta^{-1}U \tag{3a}$$

$$M(\zeta) = -2\mathrm{i}\zeta^{-4}\sigma_3 + 2\zeta^{-3}U - \mathrm{i}\zeta^{-2}U^2\sigma_3 - \zeta^{-1}(\mathrm{i}U_x\sigma_3 - U^3) \tag{3b}$$

$$U = \begin{pmatrix} 0 & u(xt) \\ -\overline{u(xt)} & 0 \end{pmatrix} \tag{4}$$

and the overbar denotes the complex conjugate.

We first use the PM method to find the one-soliton DM of the DNLS equation. To do this, we define a one-parameter transformation for (2)

$$F_j(\zeta) = T_j(\zeta)F_{j-1}(\zeta) \tag{5}$$

where the subscript j of $F_j(\zeta)$ denotes the solution of (2) with j parameters. The derivation of the explicit form of $T_j(\zeta)$ is exactly the same as that of the NLS equation [5] and the result is

$$T_j(\zeta) = I + \frac{\zeta_j - \bar{\zeta}_j}{\zeta - \zeta_j} P_j(\zeta_j) \tag{6}$$

where ζ_j is parameter and P_j is a projection matrix ($P_j^2 = P_j$, $P_j^+ = P_j$) which is determined by F_{j-1}:

$$P_j(\zeta_j) = VV^\dagger / V^\dagger V \tag{7}$$

$$V^\dagger = (b_j, 1) F_{j-1}^\dagger(\bar{\zeta}_j) \tag{8}$$

where the dagger denotes the Hermitian conjugate and b_j is an arbitrary constant. For the NLS equation, the one-parameter transformation is just its one-soliton DM [5]. However, as each soliton of the DNLS equation is determined by two parameters [6, 7], we need a two-parameter transformation which is defined as

$$F_{j+1}(\zeta) = D_{j+1}(\zeta) F_{j-1}(\zeta) \tag{9}$$

$$D_{j+1}(\zeta) = T_{j+1}(\zeta) T_j(\zeta). \tag{10}$$

From (7), it is easy to prove that

$$P_{j+1}(\zeta_{j+1}) = T_j(\bar{\zeta}_{j+1}) P_j(\zeta_{j+1}) T_j^\dagger(\bar{\zeta}_{j+1}) / \mathrm{tr}[T_j(\bar{\zeta}_{j+1}) P_j(\zeta_{j+1}) T_j^+(\bar{\zeta}_{j+1})]. \tag{11}$$

Using (11), $D_{j+1}(\zeta)$ can be expressed as in terms of P_j

$$D_{j+1}(\zeta) = I + \frac{\zeta_j - \bar{\zeta}_j}{\zeta - \zeta_j} B_1 + \frac{\zeta_{j+1} - \bar{\zeta}_{j+1}}{\zeta - \zeta_{j+1}} B_2 \tag{12}$$

$$B_1 = \left(I + \frac{\zeta_{j+1} - \bar{\zeta}_{j+1}}{\bar{\zeta}_j - \zeta_{j+1}} P_j(\zeta_{j+1}) \right) P_j(\zeta_j) / N \tag{13}$$

$$N = I + \frac{(\zeta_j - \bar{\zeta}_j)(\zeta_{j+1} - \bar{\zeta}_{j+1})}{(\zeta_j - \bar{\zeta}_{j+1})(\bar{\zeta}_j - \zeta_{j+1})} \mathrm{tr}[P_j(\zeta_j) P_j(\zeta_{j+1})]. \tag{14}$$

B_2 can be obtained by the interchange of ζ_{j+1} and ζ_j in B_1.

The one-soliton DM of the DNLS equation is the special case of the two-parameter transformation with $\zeta_{j+1} = -\zeta_j$:

$$P_j(-\zeta_j) = \sigma_3 P_j(\zeta_j) \sigma_3 \qquad B_2 = \sigma_3 B_1 \sigma_3. \tag{15}$$

These conditions come from the following properties of $L(\zeta)$ and $M(\zeta)$ [6]:

$$L(-\zeta) = \sigma_3 L(\zeta) \sigma_3 \qquad M(-\zeta) = \sigma_3 M(\zeta) \sigma_3. \tag{16}$$

For the DNLS equation, it is better to relabel the one-soliton DM as

$$F_n(\zeta) = D_n(\zeta) F_{n-1}(\zeta) \tag{17}$$

$$D_n(\zeta) = I + \frac{\zeta_n - \bar{\zeta}_n}{\zeta - \zeta_n} B_n - \frac{\zeta_n - \bar{\zeta}_n}{\zeta + \zeta_n} \sigma_3 B_n \sigma_3 \tag{18}$$

$$B_n = \left(I - \frac{\zeta_n - \bar{\zeta}_n}{\zeta_n + \bar{\zeta}_n} \sigma_3 P_n(\zeta_n) \sigma_3 \right) P_n(\zeta_n) / N \tag{19}$$

$$N = I - \frac{(\zeta_n - \bar{\zeta}_n)^2}{(\zeta_n + \bar{\zeta}_n)^2} \mathrm{tr}[P_n(\zeta_n) P_n(-\zeta_n)] \tag{20}$$

where the subscript n denotes the n-soliton solution:

$$F_n(\zeta) = F_{j+1}(\zeta) \qquad F_{n-1}(\zeta) = F_{j-1}(\zeta) \qquad D_n(\zeta) = D_{j+1}(\zeta)$$
$$\zeta_n = \zeta_j \qquad P_n(\zeta_n) = P_j(\zeta_j) \qquad B_n = B_1. \tag{21}$$

Since $P_n(\zeta_n)$ can be written as

$$P_n(\zeta_n) = \begin{pmatrix} |f_n|^2 & f_n \\ \bar{f}_n & 1 \end{pmatrix} (1 + |f_n|^2)^{-1} \tag{22}$$

where

$$f_n = v_n / w_n \qquad \begin{pmatrix} v_n \\ w_n \end{pmatrix} = V \tag{23}$$

we have

$$B_n = \frac{\zeta_n + \bar{\zeta}_n}{2} \begin{pmatrix} \dfrac{|f_n|^2}{\bar{\zeta}_n + \zeta_n |f_n|^2} & \dfrac{f_n}{\bar{\zeta}_n + \zeta_n |f_n|^2} \\ \dfrac{\bar{f}_n}{\zeta_n + \bar{\zeta}_n |f_n|^2} & \dfrac{1}{\zeta_n + \bar{\zeta}_n |f_n|^2} \end{pmatrix}. \tag{24}$$

Equation (24) is the same as that given by the modified method [6]. It must be pointed out that (12) is also the two-parameter transformation matrix of the NLS equation and in this context it is nothing but the two-soliton DM [8]. Thus the NLS and DNLS equations can be treated in a unified way.

Finally, we derive a Backlund transformation for the DNLS equation. Substituting (17) into (2), we find

$$D_{nx}(\zeta) = L_n(\zeta) D_n(\zeta) - D_n(\zeta) L_{n-1}(\zeta). \tag{25}$$

Using (18) and setting $\zeta = \zeta_n$, $\zeta = -\zeta_n$ and $\zeta = \bar{\zeta}_n$, respectively, (25) gives

$$B_{nx} = L_n(\zeta_n) B_n - B_n L_{n-1}(\zeta_n) \tag{26}$$

$$\sigma_3 B_{nx} \sigma_3 = L_n(-\zeta_n) \sigma_3 B_n \sigma_3 - \sigma_3 B_n \sigma_3 L_{n-1}(-\zeta_n) \tag{27}$$

$$-\left(B_{nx} + \frac{\zeta_n - \bar{\zeta}_n}{\zeta_n + \bar{\zeta}_n} \sigma_3 B_{nx} \sigma_3 \right) = L_n(\bar{\zeta}_n) D_n(\bar{\zeta}_n) - D_n(\bar{\zeta}_n) L_{n-1}(\bar{\zeta}_n). \tag{28}$$

Eliminating B_{nx} from the three equations above, we get

$$U_n = D_n(0) U_{n-1} D_n^\dagger(0) - 2\mathrm{i} \frac{\zeta_n - \bar{\zeta}_n}{\zeta_n^2} [\sigma_3, B_n] D_n^\dagger(0). \tag{29}$$

The matrix elements of (29) give

$$u_n = u_{n-1} \left(\frac{\zeta_n + \bar{\zeta}_n |f_n|^2}{\bar{\zeta}_n + \zeta_n |f_n|^2} \right)^2 - 2\mathrm{i} \frac{\zeta_n^2 - \bar{\zeta}_n^2}{\zeta_n \bar{\zeta}_n} \frac{f_n (\zeta_n + \bar{\zeta}_n |f_n|^2)}{(\bar{\zeta}_n + \zeta_n |f_n|^2)^2}. \tag{30}$$

Equation (29) or (30) is the Backlund transformation for the DNLS equation.

References

[1] Levi D, Ragnisco O and Sym A 1982 *Lett. Nuovo Cimento* **33** 401

[2] Levi D, Ragnisco O and Sym A 1984 *Nuovo Cimento* B **83** 34

[3] Levi D, Ragnisco O and Sym A 1984 *Phys. Lett.* **100A** 7
[4] Neugebauer G and Meinel R 1984 *Phys. Lett.* **100A** 467
[5] Chen Z Y, Huang N N and Xiao Y 1988 *Phys. Rev.* A **38** 4355
[6] Huang N N and Chen Z Y 1990 *J. Phys. A: Math. Gen.* **23** 439
[7] Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798
[8] Xiao Y 1989 *Acta Phys. Sin.* **38** 1911